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Negative Absorption by the Electrons in a Magnetic Trap

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Abstract

The absorption of the high-frequency waves by a non-relativistic electron moving in a magnetic trap is considered in the dipolar approximation. The domains in which negative absorption takes place are found.

The existence of negative cyclotron absorption in an homogeneous and constant magnetic field is well known (Schneider, 1959; Sokolov & Ternov, 1966; Gaponov & Petelin, 1967). This effect is due to the relativistic corrections, and is thus rather small. Here, we shall consider the more complicated case of an inhomogeneous magnetic field having the potential

$$\mathbf{A}_c = B_0\left\{-\frac{y}{2}\left(1+\frac{z^2}{\Delta^2}\right);\ \frac{x}{2}\left(1+\frac{z^2}{\Delta^2}\right);\ 0\right\} \tag{1}$$

The motion in such a field is finite and characterized by three frequencies. Upon certain combinations of these frequencies negative absorption will take place, even in the non-relativistic case. For each of these combinations, this effect exists only for certain conditions imposed on the relation between the oscillatory and the rotational parts of energy.

We will first examine the motion of the particle in the magnetic field (1) in the absence of an external electromagnetic field of high frequency. We will then develop a formalism for calculating the absorption power, valid in the non-relativistic and dipolar approximation, which can also be used for many other problems. We will finally find the zones of positive and negative absorption and study their dependence on the oscillatory–rotational energy ratio.

1. *Unperturbed Motion*

For our problem it is convenient to use cylindrical coordinates, in which the Hamiltonian is

$$H_0 = \frac{1}{2m}\left[P_r^2 + P_z^2 + \left(\frac{P_\phi}{r} - eA_\phi\right)^2\right]; \qquad A_\phi = \frac{rB_0}{2}\left(1+\frac{z^2}{\Delta^2}\right);$$

$$A_r = A_z = 0 \tag{1.1}$$

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The range of rotation of the electron changes according to its displacement along the z axis. In the position $z = 0$ the electron describes a circle of radius

$$r_0 = \left(\frac{2cP_\phi}{|e|B_0}\right)^{1/2}$$

in an arbitrary position its radius of rotation can be represented as $r = r_0(1 + \xi)$. Let us assume that the possible values of ξ are small and that the parameter of inhomogeneity Δ is large in comparison to the amplitude of the axial oscillations: $z/\Delta \ll 1$. Under such conditions we can rewrite (1.1) in the following manner:

$$H_0 = \frac{1}{2m}\left[P_r^2 + P_z^2 + \frac{2}{r_0^2}P_\phi^2\left(1 + \xi^2 + \frac{z^2}{\Delta^2}\right)\right] \tag{1.2}$$

The corresponding Hamilton-Jakoby equation is

$$2mH_0 = \frac{1}{r_0^2}\left(\frac{\partial S}{\partial \xi}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 + \frac{2}{r_0^2}\left(\frac{\partial S}{\partial \varphi}\right)^2\left(1 + \xi^2 + \frac{z^2}{\Delta^2}\right) \tag{1.3}$$

Its solution will be of the form

$$S = P_\phi . \varphi + S_1(\xi) + S_2(z)$$

Since the movement of the electron is finite and periodic it is reasonable to introduce the action-angle variables $I_i - \mathscr{W}_i$. Hence, the solution of the equation (1.3) expressed in terms of I_i, ξ, φ, z is written in the following manner:

$$S = I_\phi . \varphi + I_z\left(\frac{z}{z_0}\sqrt{\left[1 - \frac{z^2}{z_0^2}\right]} + \arcsin\frac{z}{z_0}\right) + I_\xi\left(\frac{\xi}{\xi_0}\sqrt{\left[1 - \frac{\xi^2}{\xi_0^2}\right]} + \arcsin\frac{\xi}{\xi_0}\right) \tag{1.4}$$

where $z_0 = (I_z r_0 \Delta / I_\phi)^{1/2}$; $\xi_0 = (I_\xi / I_\phi)^{1/2}$; and the Hamiltonian (1.2) is equal to

$$H_0 = \omega_H\left(I_\xi + I_\phi + \frac{r_0}{\Delta}I_z\right); \qquad \omega_H = \frac{|e|\,B_0}{mc} \tag{1.5}$$

We can now express the coordinates of the electron in terms of the angular variables $\mathscr{W}_\phi$, $\mathscr{W}_z$, $\mathscr{W}_\xi$ according to the equations $\mathscr{W}_i = \partial S / \partial I_i$.

It will be shown that

$$\varphi = \mathscr{W}_\phi - \frac{I_z}{4I_\phi}\sin 2\mathscr{W}_z - \frac{I_\xi}{2I_\phi}\sin 2\mathscr{W}_\xi$$

$$z = z_0 \sin\mathscr{W}_z; \qquad \xi = \xi_0 \sin\mathscr{W}_\xi \tag{1.6}$$

In addition, the dependence on time of the variables according to Hamilton's equations $\mathscr{W}_i = \partial H_0/\partial I_i$ is that of uniform motion:

$$\mathscr{W}_i = \omega_i t + \delta_i; \qquad \omega_i = \frac{\partial H_0}{\partial I_i}$$

$$\omega_\phi = \omega_H\left(1 + \frac{r_0}{2\Delta}\frac{I_z}{I_\phi}\right); \qquad \omega_\xi = \omega_H; \qquad \omega_z = \omega_H \frac{r_0}{\Delta} \tag{1.7}$$

In accordance with our assumptions, we have

$$\frac{r_0 I_z}{2\Delta I_\phi} = \frac{z_0^2}{2\Delta^2} \ll 1$$

and, consequently, $\omega_\phi \approx \omega_\xi = \omega_H$. The admissible values of the ratio I_z/I_ϕ depend on the relation ω_z/ω_H. If $\omega_z \sim \omega_H$, the energy of the axial oscillations must be relatively small, that is, $I_z/I_\phi \ll 1$. For $\omega_z \ll \omega_H$, the ratio I_z/I_ϕ can be of the order of, or greater than, 1. We can see from the formula (1.5) that the phasal motion consists of three parts: a uniform motion of the 'velocity' ω_ϕ; an oscillatory motion caused by the axial motion; and an oscillatory motion due to the radial oscillations. According to the assumption $\xi_0 \ll 1$, the last term is always small; this is not true for the second term when $r_0/\Delta \ll 1$. For this reason the second term must not be omitted. Therefore, by introducing the complex coordinate $x + iy$, we will have

$$\begin{aligned} x + iy = \sum_{n=-\infty}^{\infty} r_0\, J_n\left(\frac{I_z}{4I_\phi}\right) \Big\{ &\exp i(\mathscr{W}_\phi - 2n\mathscr{W}_z) \\ &+ \frac{\xi_0}{2i}[\exp i(\mathscr{W}_\phi - 2n\mathscr{W}_z + \mathscr{W}_\xi) - \exp i(\mathscr{W}_\phi - 2n\mathscr{W}_z - \mathscr{W}_\xi)] \\ &- \frac{\xi_0^2}{4}[\exp i(\mathscr{W}_\phi - 2n\mathscr{W}_z + 2\mathscr{W}_\xi) \\ &- \exp i(\mathscr{W}_\phi - 2n\mathscr{W}_z - 2\mathscr{W}_\xi)] + \dots \Big\} \end{aligned} \tag{1.8}$$

From this formula it is clear that the most important contribution in the processes of absorption and emission is given by the 'principal series' of resonances having the frequency $\omega_n = \omega_\phi - 2n\omega_z$.

2. *Interaction with Electromagnetic Waves*

The power transferred to the electron by a variable electromagnetic field can be calculated as the amount of work per second created by the forces of the field on the trajectory perturbed by the field itself. The method to be developed here is in fact a generalization of the method utilized by Sobelman & Tutin (1963) for the case of a polyperiodic oscillatory motion.

Write the potential due to the free oscillations of the field in the form

$$\mathbf{A}^{\sim} = \sum_{\mu} a_\mu \mathbf{e}_\mu \cos(\omega_\mu t + \varphi_\mu); \qquad |\mathbf{e}_\mu| = 1 \tag{2.1}$$

This means that we will hereafter be limited by the dipolar approximation. From the above it is clear that the power absorbed by the electron is equal to

$$W = -\frac{e}{c}\langle \dot{\mathbf{r}}\dot{\mathbf{A}}^{\sim}\rangle \tag{2.2}$$

where the average must be taken over phases and time. In this formula $\dot{\mathbf{r}}$ is the velocity of the perturbed motion. If the ratio $A^{\sim}/A_c$ is small, this velocity can be represented in the form of a series: $\dot{\mathbf{r}} = \dot{\mathbf{r}}^{(0)} + \dot{\mathbf{r}}^{(1)} + \dot{\mathbf{r}}^{(2)} + \cdots$. The first non-vanishing contribution is given by the term $\dot{\mathbf{r}}^{(1)}$, because $\dot{\mathbf{r}}^{(0)}$ does not depend on the variable field, and thus disappears in the process of taking the average. Let us now pass to the calculation of $\dot{\mathbf{r}}^{(1)}(t)$. The Hamiltonian of the system electron + field will be the following:

$$H = \frac{1}{2m}\left[\mathbf{P} - \frac{e}{c}\mathbf{A}_c - \frac{e}{c}\mathbf{A}^{\sim}\right]^2 = H_0 - \frac{e}{mc}\left(\mathbf{P} - \frac{e}{c}\mathbf{A}_c\right)\mathbf{A}^{\sim} + \frac{e^2}{2mc^2}\mathbf{A}^{\sim 2} \tag{2.3}$$

We can omit the last term in (2.3), since its contribution to the absorption disappears on averaging over δ_i. Supposing that the interaction between the field and the particles begins at the moment $t = 0$, we can find the variations

$$\Delta I_i = -\int_0^t \frac{\partial H}{\partial \mathscr{W}_i} \qquad \text{and} \qquad \Delta \mathscr{W}_i = \int_0^t \frac{\partial H}{\partial I_i}$$

resulting from it. It then becomes easy to calculate $\dot{\mathbf{r}}^{(1)}$:

$$\dot{\mathbf{r}}^{(1)} = \sum_i \left(\Delta I_i \frac{\partial \dot{\mathbf{r}}^{(0)}}{\partial I_i} + \Delta \mathscr{W}_i \frac{\partial \dot{\mathbf{r}}^{(0)}}{\partial \mathscr{W}_i}\right) \tag{2.4}$$

Substituting (2.4) in (2.2) we have the following expression for absorption:

$$W = +\frac{e^2}{2c^2}\sum_{\mu} \lim_{T\to\infty}\frac{1}{T}\int_0^T dt \int_0^t dt'\, a_\mu^2 \omega_\mu \Big\langle \sin\omega_\mu(t'-t)\{\mathbf{e}_\mu \dot{\mathbf{r}}^{(0)}(t); \mathbf{e}_\mu \dot{\mathbf{r}}^{(0)}(t')\}$$

$$+ \frac{\partial \omega_i}{\partial I_j}\int_0^{t'} \sin\omega_\mu(t''-t)\frac{\partial \mathbf{e}_\mu \dot{\mathbf{r}}^{(0)}(t'')}{\partial \mathscr{W}_j}\cdot\frac{\partial \mathbf{e}_\mu \dot{\mathbf{r}}^{(0)}(t)}{\partial \mathscr{W}_i}\Big\rangle \tag{2.5}$$

Here { } represents the classic Poisson brackets. The summation over ij is understood. Finally, introducing the spectral density of the intensity $I_{\mathbf{k},\lambda}$ as the characteristic of the incident radiation, we obtain

$$dW = +\frac{4\pi e^2}{\omega c}\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\int_0^t dt'\sum_\lambda\Bigg[\sin\omega(t'-t)\langle\{\mathbf{e}_\lambda\dot{\mathbf{r}}^{(0)}(t);\mathbf{e}_\lambda\mathbf{r}^{(0)}(t')\}\rangle$$

$$+\frac{\partial\omega_i}{\partial I_j}\int_0^{t'}\sin\omega(t''-t)\left\langle\frac{\partial\mathbf{e}_\lambda\dot{\mathbf{r}}^{(0)}(t'')}{\partial\mathscr{W}_j}\cdot\frac{\partial\mathbf{e}_\lambda\dot{\mathbf{r}}^{(0)}(t)}{\partial\mathscr{W}_i}\right\rangle\Bigg]I_{\mathbf{k},\lambda}\,d\omega\,dO \qquad (2.6)$$

λ is the polarization index.

3. *Negative Absorption at the Frequencies of the 'Principal Series'*

Let us suppose that the incidental wave is polarized along the y-axis. Substituting (1.8) in (2.6) and taking all the necessary averages we have:

$$dW = \frac{4\pi^3 e^2}{\omega c}\sum_n\Bigg\{\hat{D}_n\left[r_0\,J_n\!\left(\frac{I_z}{4I_\phi}\right)\omega_n\right]^2[\delta(\omega-\omega_n)-\delta(\omega+\omega_n)]$$

$$-(\hat{D}_n\omega_n)\left[r_0\,J_n\!\left(\frac{I_z}{4I_\phi}\right)\omega_n\right]^2[\delta'(\omega-\omega_n)+\delta'(\omega+\omega_n)]\Bigg\}I(\omega)\,d\omega \quad (3.1)$$

$$\hat{D}_n = \frac{\partial}{\partial I_\phi} - 2n\frac{\partial}{\partial I_z}$$

The number of terms in the sum over n is determined by the effective width of the spectral function $I(\omega)$ compared to the difference

$$\omega_n - \omega_{n-1} = 2\frac{\omega_H r_0}{\Delta}$$

If the exterior radiation possesses a sufficiently narrow spectrum, the δ functions contribute only once. Integrating (3.1) over ω we have the following expression for the absorption of the nth harmonic:

$$W_n = \frac{4\pi^3 e^2}{c}\hat{D}_n\left[r_0^2\,\omega_n\,J_n^2\!\left(\frac{I_z}{4I_\phi}\right)I(|\omega_n|)\right] \qquad (3.2)$$

where the derivative of the spectral density is defined by

$$\hat{D}_n I(|\omega_n|) = \frac{\partial I}{\partial\omega}(\hat{D}_n\omega_n)\,\text{sign}\,\omega_n$$

Let us now examine the sign of the power (3.2). Having completed all the necessary differentiations in the formula (3.2), we arrive at the following:

$$W_n = \frac{8\pi^3 e^2}{mc} J_n^2 \left\{ \left[1 - 4n\frac{r_0}{\Delta} - \frac{1}{2}\frac{\omega_n}{\omega_H}\frac{J_n}{J_n'}\left(2n + \frac{I_z}{I_\phi}\right)\right] I(|\omega_n|) - |\omega_n| \frac{\partial I}{\partial \omega}\bigg|_{\omega=|\omega n|} \cdot \frac{r_0}{\Delta}\left(2n + \frac{1}{4}\frac{I_z}{I_\phi}\right)\right\} \tag{3.3}$$

It is necessary to notice that the limit of possible applicability of this formula is the condition

$$\frac{r_0 I_z}{\Delta I_\phi} = \frac{z_0{}^2}{\Delta^2} \ll 1$$

This small parameter represents the product of the ratio r_0/Δ, characterizing the degree of inhomogeneity of the magnetic field, with the ratio I_z/I_ϕ characterizing the degree of excitation of the axial oscillation. First of all, consider the case of weak excitation of z oscillations, that is $I_z \ll I_\phi$. In this instance W_n is visibly distinct from zero only for $n = 0, \pm 1$. Instead of Bessel-functions, the approximate expressions

$$J_n(z) \simeq (n!)^{-1}\left(\frac{z}{2}\right)^n$$

can be used. Then, for $n = 0$,

$$W_0 = \frac{8\pi^3 e^2}{mc}\left(I(\omega_\phi) - \omega_\phi \frac{r_0 I_z}{4\Delta I_\phi}\frac{\partial I}{\partial \omega}\bigg|_{\omega\phi}\right) \tag{3.4}$$

Because of the condition $z_0 \ll \Delta$ we can omit the second term, thus arriving at the well-known formula for cyclotron absorption (Sokolov & Ternov, 1966). For the harmonics $n = \pm 1$ in the same approximation, we obtain the following formula:

$$W_{\pm 1} = \mp \frac{8\pi^3 e^2}{mc}\frac{I_z}{I_\phi}\left\{\left[1 \mp 2\frac{r_0}{\Delta} \pm \frac{1}{4}\frac{I_z}{I_\phi}\left(1 \pm 2\frac{r_0}{\Delta}\right)\right] I(|\omega_{\pm 1}|) + \frac{1}{2}\frac{r_0}{\Delta}\frac{I_z}{I_\phi}|\omega_{\pm 1}|\frac{\partial I}{\partial \omega}\bigg|_{\omega=|\omega\pm 1|}\right\} \tag{3.5}$$

As in the preceding case, the influence of the second term is negligible. It is easy to see that, for $n = -1$, absorption occurs for any value of the ratio r_0/Δ. For the frequency ω_1 with the condition

$$\Delta > 2r_0 \tag{3.6}$$

the sign of the absorption becomes negative, that is, amplification of the waves takes place. It is necessary, however, to notice that this effect is I_z/I_ϕ times less than the absorption, upon the frequency ω_0.

Now examine the case when the parameter of inhomogeneity is small, $r_0/\Delta \ll 1$. In this case, the condition $z_0 \ll \Delta$ does not necessarily demand that the ratio I_z/I_ϕ be small. Consider it as being comparable to unity. In these conditions only those n for which $(r_0/\Delta)n \ll 1$ are of interest to us. (In the opposite case $\mathcal{T}_n(I_z/4I_\phi) \ll 1$.) Then from (3.3) we find the following condition of negative absorption:

$$J_n^2(z) - (n + 2z)\, J_n'(z)\, J_n(z) - \frac{r_0\, \omega_H}{\Delta} . \frac{\partial}{\partial \omega} \ln I(\omega) . (2n + z)\, J_n^2(z) < 0$$

$$z = I_z/4I_\phi$$

If the maximum of spectral intensity occurs near to the resonance, the last term can be neglected and our condition becomes

$$(n + 2z)\, J_n'(z)\, J_n(z) > J_n^2(z) \tag{3.7}$$

In so far as $J_n(z)$ is an oscillating function, for each n a certain domain of z can be found where this inequality is fulfilled. In fact, designate the first root of $J_n(z)$ as z_{n1}, the first root of $J_n'(z)$ as z'_{n1}, and the first root of the equation $(n + 2z)\, J_n'(z) - J_n(z) = 0$ as $\tilde{z}_{n1}$. It is then easy to see that

$$\tilde{z}_{n1} < z'_{n1} < z_{n1}, \qquad n > 0$$

The inequality (3.7) will thus be satisfied when $z < \tilde{z}_{n1}$. For $n = 0$ the condition (3.7) becomes

$$J_1(z)\, J_0(z) - J_0^2(z) > 0$$

and possesses an infinite number of solutions.

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